

# Maximum principle for a size-structured model of forest and carbon sequestration management

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Received 14 February 2007; received in revised form 26 October 2007; accepted 11 December 2007

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## Abstract

The paper analyzes nonlinear optimal control of integral–differential equations that describe the optimal management of a forest taking into account intra-species competition and carbon sequestration. The objective function includes the revenues from timber production, operational expenses, and the net benefits from carbon sequestration. A dual system is derived and a necessary extremum condition is established.

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**Keywords:** Optimal control; Nonlinear integral–differential equations; Size-structured population models

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## 1. Introduction

Age-structured population models are a traditional tool in biological modeling [1,2,9]. Recently, size-structured models have attracted large attention because they better describe the dynamics of some populations, e.g., trees, fish [7,8], and consider relevant size-dependent economic parameters. We analyze a realistic forest management problem that determines the optimal logging regime for maximizing the net benefits from a timber production with additional net benefits from carbon sequestration. Its detailed description and numeric analysis are provided in [3]. The model involves a *nonlocal* integral equation for the carbon concentration in soil and timber, which does not allow applying all previously presented results. There is no standard optimization technique for such models and their investigation is highly nontrivial. Using general optimization theory, the present paper establishes a necessary extremum condition in the form of a maximum principle (Section 2) and discusses its relevance in Section 3.

The *optimization problem* (OP) is to find functions  $u(t, l)$ ,  $p(t)$ ,  $x(t, l)$ ,  $b(t)$ ,  $s(t)$ ,  $E(t)$ ,  $V(t)$ , and  $W(t)$ ,  $l \in [l_0, l_m]$ ,  $t \in [0, T)$ ,  $T \leq \infty$ , that maximize

$$J = \int_0^T e^{-rt} \left\{ \int_{l_0}^{l_m} B(x(t, l), u(t, l)) dl + \rho_2(t) \left[ \frac{db(t)}{dt} + \frac{ds(t)}{dt} \right] - \rho_3(t) p(t) \right\} dt \quad (1)$$

under restrictions

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$$\frac{\partial x(t, l)}{\partial t} + \frac{\partial [g(E(t), l)x(t, l)]}{\partial l} = -\mu(E(t), l)x(t, l) - u(t, l), \quad t \in [0, T], l \in [l_0, l_m], \quad (2)$$

$$\frac{ds(t)}{dt} = h \left( \frac{dV(t)}{dt}, s(t) \right), \quad t \in [0, T], s(0) = s_0, \quad (3)$$

$$V(t) = \gamma_0 \int_{l_0}^{l_m} l^\beta x(t, l) dl, \quad W(t) = \frac{dV(t)}{dt}, \quad (4)$$

$$E(t) = \chi \int_{l_0}^{l_m} l^2 x(t, l) dl, \quad b(t) = \gamma_0 \int_{l_0}^{l_m} v(l) l^\beta x(t, l) dl, \quad v'(l) > 0, \quad (5)$$

$$x(0, l) = x_0(l), \quad l \in [l_0, l_m], \quad x(t, l_0) = p(t), \quad t \in [0, T], \quad (6)$$

$$0 \leq u(t, l) \leq u_{\max}(t, l), \quad 0 \leq p(t) \leq p_{\max}(t). \quad (7)$$

In (1)–(7),  $l$  denotes the diameter of a tree,<sup>1</sup>  $l \in [l_0, l_m]$ , where  $l_0$  is the diameter of a planted tree and  $l_m$  is its maximum diameter,  $x(t, l)$  is the distribution density of trees,  $u(t, l)$  is the flux of logged trees,  $p(t)$  is the flux of new trees planted at time  $t$  with diameter  $l_0$ ,  $g(E(t), l)$  is the growth rate of trees,  $\mu(E(t), l)$  is the instantaneous mortality rate,  $E(t)$  measures the forest density which in turn affects the growth rate of individual trees (intra-species competition),  $V(t)$  is the above-ground volume of the forest biomass,  $b(t)$  and  $s(t)$  represent the amount of carbon sequestered in the long run in the timber and soil. The condition  $v'(l) > 0$  in (5) reflects the fact that the amount of carbon sequestered in the long run is higher for larger trees than for smaller trees.<sup>2</sup> The positive parameters  $\chi$ ,  $\beta$  and  $\gamma_0$  in Eqs. (4) and (5) have to be chosen according to tree species and empirical data at hand.

Functional (1) includes the net benefits from timber production (1st term) and carbon sequestration (2nd term) minus the expenses for planting new trees (3rd term). Eq. (2) is a size-structured version of the well-known Gurtin–MacCamy model for a completely managed forest with no biological reproduction (i.e., all young trees are planted). Eqs. (3)–(5) describe the dynamics of the carbon content in the forest ecosystem.

## 2. Main results

Let us choose  $u$  and  $p$  as *independent* control variables of the OP (1)–(7). The rest of the unknown functions  $x$ ,  $E$ ,  $s$ ,  $V$ ,  $b$ , and  $W$  are *dependent* (state) variables. For convenience, we will use only two “essential” state variables  $x(t, l)$  and  $q(t) = s'(t)$ . All other state variables can be expressed through them and  $u$  and  $p$  by (3)–(5).

We assume that  $u \in L_\infty([0, T] \otimes [l_0, l_m])$ ,  $p \in L_\infty[0, T]$ . The first step of an optimization technique for such OPs [1,4,9] is to define the solution  $(x, q)$  of the state equations (2)–(5) under given controls  $u$  and  $p$  and establish conditions on the given functions when such a solution exists. The corresponding reasoning is standard (e.g., [1,2,5,6,9]) but is beyond the scope of this paper due to space limitations. Instead, we assume that the given  $g$ ,  $\mu$ ,  $B$ ,  $\rho_2$ ,  $\rho_3$ ,  $h$ ,  $v$ ,  $x_0$  are smooth and “good enough” for a unique solution  $(x, q)$  to exist in the weak sense defined in [1,2,9] for age-structured models and in [6] for size-structured models.

**Remark.** Some of the state variables, in particular  $x$ , must be nonnegative for the OP to have economic sense. As in other similar investigations, we assume that the positiveness requirements are satisfied automatically.

The next step is to prove the differentiability of the OP and find its functional derivatives. For OPs with equalities-constraints, this step involves a dual system. We introduce the *dual (adjoint) system* for the OP (1)–(7) as

$$\begin{aligned} \frac{\partial \lambda(t, l)}{\partial t} + g(E(t), l) \frac{\partial \lambda(t, l)}{\partial l} &= [r + \mu(E(t), l)] \lambda(t, l) - B_x(x(t, l), u(t, l)) + \chi l^2 \gamma(t) \\ &+ F_t(t, l) - r F(t, l), \end{aligned} \quad (8)$$

<sup>1</sup> As established by forest scientists, the age of a tree is not closely related to its size. A large genetic variety between the trees and of the different local conditions of each tree makes it difficult to establish a functional relationship between size and age. Consequently, the age–size relationship resulting from an econometric estimation is very poor.

<sup>2</sup> The carbon content is more or less constant throughout the growth process of the tree. However, the larger is the tree the more valuable is the timber. Therefore it is more likely that it will be used for durable goods (furniture) and not for disposable goods (paper). Consequently, the amount of carbon sequestered in the long run increases with the size of the tree.

$$\lambda(T, l) = 0, \quad l \in [l_0, l_m], \quad \lambda(t, l_m) = 0, \quad t \in [0, T], \quad (9)$$

$$\zeta(t) - \int_t^T e^{-r(\tau-t)} \frac{\partial}{\partial s} h \left( W(\tau), s_0 + \int_0^\tau q(u) du \right) \zeta(\tau) d\tau = \rho_2(t), \quad (10)$$

where  $\lambda(t, l)$  and  $\zeta(t)$  are unknown dual variables and

$$F(t, l) = \gamma_0 l^\beta \left[ \rho_2(t) v(l) + \zeta(t) \frac{\partial}{\partial W} h \left( W(t), s_0 + \int_0^t q(u) du \right) \right], \quad (11)$$

$$\gamma(t) = \int_{l_0}^{l_m} \left\{ -\frac{\partial [g_E(E(t), l) x(t, l)]}{\partial l} - \mu_E(E(t), l) x(t, l) \right\} \lambda(t, l) dl. \quad (12)$$

Here and thereafter we use the common notation  $f_x$  for  $\partial f / \partial x$ . The dual system (8)–(10) is linear and the conditions of its solvability are less restrictive than for the state equations (2)–(5). The condition  $v'(l) > 0$  in (5) reflects the fact that the amount of carbon stored over a long time horizon is higher for larger trees.

The controls  $u(t, l)$ ,  $p(t)$ ,  $t \in [0, T]$ ,  $l \in [l_0, l_m]$ , are *admissible* if they and the corresponding state variables satisfy the OP restrictions (2)–(7). The variations  $\delta u(t, l)$ ,  $\delta p(t)$ ,  $t \in [0, T]$ ,  $l \in [l_0, l_m]$ , *admissible* if functions  $u + \delta u$ ,  $p + \delta p$ ,  $u$ , and  $p$  are admissible.

**Theorem 1** (On the OP Differentiability.). *If the state equations (2)–(5) and dual equations (8)–(10) have unique solutions under given controls  $u$  and  $p$ , then the functional (1) is differentiable and for any admissible variations  $\delta u$ ,  $\delta p$ , the increment  $\delta J$  of the functional  $J$  is of the form:*

$$\delta J = \int_0^T e^{-rt} \left\{ \int_{l_0}^{l_m} \frac{\partial J}{\partial u}(t, l) \delta u(t, l) dl + \frac{\partial J}{\partial p}(t) \delta p(t) \right\} dt + \delta^2 I, \quad \delta^2 I = o(\|\delta u\|, \|\delta p\|), \quad (13)$$

where the Freshet derivatives of the functional  $J$  in  $u$  and  $p$  are:

$$\frac{\partial J}{\partial u}(t, l) = e^{-rt} [B_{1u}(x(t, l), u(t, l)) - \lambda(t, l)], \quad l \in [l_0, l_m], \quad (14)$$

$$\frac{\partial J}{\partial p}(t) = e^{-rt} [g(E(t), l_0) \lambda(t, l_0) - \rho_3(t)], \quad t \in [0, T]. \quad (15)$$

**Proof.** Let us give the admissible variations to the unknown  $u$ ,  $p$ ,  $x$ , and  $q$  and consider the corresponding variation  $\delta J$  of the functional (1):

$$\begin{aligned} \delta J &= J(u + \delta u, p + \delta p) - J(u, p) \\ &= \int_0^T e^{-rt} \left\{ \int_{l_0}^{l_m} [B_x(x(t, l), u(t, l)) \delta x(t, l) + B_u(x(t, l), u(t, l)) \delta u(t, l)] dl \right. \\ &\quad \left. + \rho_2(t) \gamma_0 \int_{l_0}^{l_m} v(l) l^\beta \frac{\partial \delta x(t, l)}{\partial t} dl + \delta q(t) - \rho_3(t) \delta p(t) \right\} dt + \delta^2 I, \quad \delta^2 I = o(\|\delta u\|, \|\delta p\|). \end{aligned} \quad (16)$$

To eliminate the terms with  $\delta x$  and  $\delta q$  in (16), we perform the following two steps.

*Step 1.* (elimination of  $\delta q$ ). Multiplying the dual equation (10) by  $e^{-rt} \delta q(t)$ , integrating the resulting formula in  $t$  over  $[0, T]$ , and interchanging the limits of integration, we obtain

$$\begin{aligned} &\int_0^T e^{-rt} \zeta(t) \delta q(t) dt - \int_0^T e^{-rt} \zeta(t) \int_0^t \delta q(\tau) d\tau \frac{\partial}{\partial s} h \left( W(t), s_0 + \int_0^t q(u) du \right) dt \\ &= \int_0^T e^{-rt} \rho_2(t) \delta q(t) dt. \end{aligned} \quad (17)$$

Combining (11), (16) and (17) with the Eq. (3) in variation of the form

$$\delta q(t) - \frac{\partial}{\partial s} h \left( W(t), s_0 + \int_0^t q(\tau) d\tau \right) \int_0^t \delta q(\tau) d\tau = \frac{\partial}{\partial W} h \left( W(t), s_0 + \int_0^t q(\tau) d\tau \right) \gamma_0 \int_{l_0}^{l_m} l^\beta \frac{\partial \delta x(t, l)}{\partial t} dl$$

we exclude the term with  $\delta q$  in (16) and obtain the following expression for  $\delta J$  (that still involves  $\delta x$ ):

$$\begin{aligned} \delta J \approx & \int_{l_0}^{l_m} \int_0^T e^{-rt} \{ [B_x(x(t, l), u(t, l)) - F_t(t, l) + rF(t, l)] \delta x(t, l) + B_u(x(t, l), u(t, l)) \delta u(t, l) \} dt dl \\ & - \int_0^T e^{-rt} \rho_3(t) \delta p(t) dt. \end{aligned} \quad (18)$$

*Step 2.* (elimination of  $\delta x$ ). Let us multiply the dual equation (8) by  $e^{-rt} \delta x(t, l)$ , integrate the resulting formula over  $[0, T] \otimes [l_0, l_m]$ , apply integration by parts, and use the boundary conditions (9). We obtain

$$\begin{aligned} & - \int_{l_0}^{l_m} \int_0^T e^{-rt} \left\{ \frac{\partial \delta x(t, l)}{\partial t} + \frac{\partial [\delta x(t, l) g(E(t, l))]}{\partial l} \right\} \lambda(t, l) dt dl - \int_0^T e^{-rt} g(E(t, l_0)) \lambda(t, l_0) \delta p(t) dt \\ & = \int_{l_0}^{l_m} \int_0^T e^{-rt} \lambda(t, l) \mu(E(t, l)) \delta x(t, l) dt dl \\ & + \int_{l_0}^{l_m} \int_0^T e^{-rt} [-B_x(x(t, l), u(t, l)) + \chi l^2 \gamma(t) + F_t(t, l) - rF(t, l)] \delta x(t, l) dt dl. \end{aligned} \quad (19)$$

Applying Eq. (13) and the Eqs. (2) and (5) in variations, we simplify (19) as

$$\begin{aligned} & \int_{l_0}^{l_m} \int_0^T e^{-rt} \lambda(t, l) \delta u(t, l) dt dl - \int_0^T e^{-rt} g(E(t, l_0)) \lambda(t, l_0) \delta p(t) dt \\ & = \int_{l_0}^{l_m} \int_0^T e^{-rt} [-B_x(x(t, l), u(t, l)) + F_t(t, l) - rF(t, l)] \delta x(t, l) dt dl. \end{aligned} \quad (20)$$

The substitution of (20) into (18) leads to the expressions (13)–(15). The theorem is proven.  $\square$

**Theorem 2** (The Maximum Principle). *Let  $(p^*, u^*, x^*, q^*)$  be a solution of the OP (1)–(7). Then*

$$\begin{aligned} \partial J / \partial u(t, l) & \leq 0 \quad \text{at } u^*(t, l) = 0, \quad \partial J / \partial u(t, l) \geq 0 \quad \text{at } u^*(t, l) = u_{\max}(t, l), \\ \partial J / \partial u(t, l) & = 0 \quad \text{at } 0 < u^*(t, l) < u_{\max}(t, l) \text{ for almost all (a.a.) } t \in [0, T], l \in [l_0, l_m]; \end{aligned} \quad (21)$$

$$\begin{aligned} \partial J / \partial p(t) & \leq 0 \quad \text{at } p^*(t) = 0, \\ \partial J / \partial p(t) & \geq 0 \quad \text{at } p^*(t) = p_{\max}(t), \quad \partial J / \partial p(t) = 0 \quad \text{at } 0 < p^*(t) < p_{\max}(t), \quad \text{a.a. } t \in [0, T]. \end{aligned} \quad (22)$$

**Proof.** Since  $(y^*, u^*, x^*, q^*)$  is a solution, then  $J(u + \delta u, p + \delta p) - J(u, p) \leq 0$  for any admissible variations  $\delta u, \delta p$ .

Let us first assume that  $\delta p \equiv 0$  and  $\delta u$  is small, then by (13)

$$J(u + \delta u, p) - J(u, p) \approx \int_0^T e^{-rt} \int_{l_0}^{l_m} \frac{\partial J}{\partial u}(t, l) \delta u(t, l) dl dt \leq 0. \quad (23)$$

The inequality (23) implies that the pointwise conditions (21) are necessary for the optimality of  $(y^*, u^*)$ . Indeed, if any of the conditions (21) is not valid, then a small admissible variation  $\delta u_c(t, l), t \in [0, T], l \in [l_0, l_m]$ , can be easily constructed such that  $J(u + \delta u_c, p) - J(u, p) > 0$  (hence,  $(y^*, u^*)$  is not a solution). The analogous reasoning in the case of  $\delta u \equiv 0$  and small  $\delta p$  leads to the necessary conditions (22). The theorem is proven.  $\square$

### 3. Summary

The obtained optimality conditions represent a key theoretic result required for further analysis of OP qualitative properties and development of efficient numeric algorithms. Owing to the structure of the domain (7) of admissible controls, the construction of a gradient projection algorithm for the OP is straightforward. A qualitative analysis of the OP is very promising and can produce such essential results as the bang-bang structure of optimal logging regimes [5]. In addition, due to the severity of future climate changes, the OP analysis provides a very helpful tool for designing climate policies in the field of forest management. However, the analytic investigation of the OP is challenging in the general case and consideration of special cases is required.

## Acknowledgements

The authors are grateful to two anonymous referees for their useful remarks. They also acknowledge the support of the NATO grant CLG982209 and Ministerio de Ciencia y Tecnología AGL2004-00964 and INIA SUM 2006-00019-C02.

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